Technical Comments

Comment on "Effect of Higher Order Terms in Certain Nonlinear Finite Element Models"

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ODEN et al.[1] conclude that convergence will be slowed if terms which characterize nonlinear behavior are approximated by a simpler polynomial than that used for the conventional stiffness matrix. The present Comment offers numerical evidence pertinent to this theoretical conclusion.

The problem chosen is of the type used in Ref. 1. A cantilever beam is modeled by ten elements, and at its free end carries an axial compressive load P and a transverse load $Q = 0.001$. Properties are chosen such that the Euler buckling load is $P_{cr} = 1.000$. The solution method is based on convected coordinates, as clearly detailed for plate problems by Murray and Wilson.[2] Thus the technique is "pure" Newton-Raphson iteration; under-relaxation, convergence-acceleration schemes, etc., were not used. Iterative cycling at a given load level was terminated by a convergence test on displacements. The next load level was then applied, and iteration begun again, starting with the displacements obtained at the previous load level.

The conventional stiffness matrix of each element (in its convected coordinate system) was based on a cubic polynomial. Two forms of 'geometric' or 'initial stress' stiffness matrix were considered.[3] The first, $[k_{GC}]$, was based on the cubic polynomial, and the second, $[k_{GL}]$, was based on a linear polynomial. In other words, with reference to Eqs. (3) of Ref. 1, $[k_{GC}]$ was based upon the lateral displacement $v(x) = \phi(x)\mathbf{v}$ and $[k_{GL}]$ upon $v(x) = \psi(x)\mathbf{v}$.

Let d/L be the ratio of lateral tip displacement to beam length. With $P = 0.99$, d/L converged to 0.0606 in 9 cycles when $[k_{GL}]$ was used, and to 0.0701 in 15 cycles when $[k_{GC}]$ was used. With the geometric stiffness matrix omitted altogether, d/L reached 0.0257 at 40 cycles but had not converged. The correct result is $d/L = 0.0811$, which is some 100 times the value produced by lateral load Q acting alone.

Table 1 lists tip rotations obtained by progressive increases in load level, with a milder convergence tolerance than used in the preceding paragraph.

Table 1 Tip rotation in radians, with cycles (in parentheses) needed for convergence in five-step loading program.

Geometric stiffness matrix	$P = 0.99$	$P = 1.05$	$P = 1.17$	$P = 1.41$	$P = 1.89$
none	0.04 ...[a]	0.35 ...[a]	1.09 (18)	1.60 (6)	2.10 (3)
$[k_{GC}]$	0.12 (7)	0.61 (22)	1.08 (12)	1.50 (8)	1.98 (13)
$[k_{GL}]$	0.10 (7)	0.62 (8)	1.10 (6)	1.61 (4)	2.11 (4)
theory[4]	...	0.62	1.09	1.59	2.10

[a] Not converged. Rotation at 40 cycles is given.

Rather than prescribing P and calculating its displacement, one may prescribe an axial displacement and calculate the corresponding P. In a loading program of six prescribed displacement levels, convergence to the correct P was roughly three times faster in each level when the geometric stiffness matrix was omitted altogether.

The foregoing numerical results conflict with the conclusions in Ref. 1. One must therefore ask whether these conclusions are not applicable to a solution based on convected coordinates, or are otherwise of a more restrictive nature than has been made clear.

References

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Reply by Authors to R. D. Cook

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THE authors wish to thank Cook for his interesting comments. We hasten to point out, however, the following:

1) The numerical results that Cook quotes have nothing to do with the subject paper; as is indicated in the opening statement of the model problem, we consider only the behavior of the structure under loads $P < P_{cr}$.

2) Even so, the results obtained by Cook can be rather easily explained, as we shall indicate below.

3) More importantly, Cook's conclusions may be misleading; for structures which are unstable at the critical load, entirely different results than those he reports could be obtained. Indeed, the particular problem Cook describes has a special property which promotes the rapid convergence mentioned in his Comment.

Concerning the first point, a check of our analysis still reveals that we assume a locally quadratic, monotone behavior in the potential energy function. That is to say, our analysis holds so long as a second variation of the energy is positive-definite. Stated in still another way, the operators entering into the nonlinear equations are strongly monotone. This is precisely why we limited our analysis to large deformations in which $P < P_{cr}$.

Now, turning to our second point, suppose we remove the assumption that $P < P_{cr}$. Then the question arises as to whether or not the structure is stable at the critical load. The sample problem picked by Cook is the classical Euler problem which is known to be stable at and beyond the critical load. This

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simply means that odd ordered variations in the energy, of higher order than two, vanish at $P = P_{cr}$.

How does this affect convergence? To answer this question, consider a (potential energy) functional $K(u)$ expanded about some function v in terms of Gateaux differentials at u_0

$$K(v) = K(u_o) + \delta K(u_o, n) + 1/2!\,\delta^2 K(u_o, n, n) + \ldots \quad (1)$$

Here $n = v - u_o$ and u, v may be vector-valued. Let u_o = exact variational solution of the problem; U = finite-element (Galerkin) approximation of the solution; and $\tilde{U}$ = an arbitrary element in the same subspace of approximants containing U. Suppose that the postbuckling path is stable. Then

$$|K(u_o) - K(U)| \leqq |K(u_o) - K(\tilde{U})| \quad (2)$$

The left-hand side of this inequality is the error in "energy." The right-hand side can be reduced using Eq. (1):

$$K(\tilde{U}) - K(u_o) = \delta K(u_o, E) + 1/2!\,\delta^2 K(u_o, E, E) + 1/3!\,\delta^3 K(u_o, E, E, E) + 1/4!\,\delta^4 K(u_o E, E, E, E) + \ldots$$

where E is the interpolation error

$$E = u_o - \tilde{U}$$

Now, since u_0 is assumed to correspond to a stable critical state, set

$$\delta K(u_o) = 0, \quad \delta^2 K(u_o) = 0, \quad \delta^3 K(u_o) = 0$$

and, for sufficiently small finite-element meshes

$$|K(u_o) - K(\tilde{U})| \leqq C_o \delta^4 K(u_o, E, E, E, E) \leqq C_1 \|E\|^4$$

where C_o and C_1 are constants. Introducing this inequality into Eq. (2) gives

$$|K(u_o) - K(\tilde{U})| \leqq C\|E\|^4$$

For consistent finite-element approximations, $\|E\| \leqq C_o h^r$, where r depends upon the degree of polynomial used in the approximation as well as the degree of the highest derivative in $K(u)$. For example, for piecewise cubics in our beam problem, $r = 4 - 2 = 2$. Notice also for our problem that when $\delta^2 K \approx 0$, $\delta^4 K$ does not depend on quadratic terms in K. We therefore conclude that: 1) the rate-of-convergence at and beyond the critical load is (at least) twice as fast as that before. Moreover, 2) the character of the initial (linear) stiffness does not influence the rate-of-convergence. These conclusions seem to be supported by Cook's experiences.

Finally, turning to item 3 mentioned earlier. It is clear from the abovementioned calculations that had the structure been unstable at the critical load, all of the preceding analysis does not hold. In such cases, the model itself may be unstable. The rate-of-convergence, if any, on points on the postbuckling path will depend upon the global geometry of the structure and its material properties, etc. Conceivably, one might also be able to consider cases in which improved convergence rates were encountered even in cases such as this. The analysis, however, is considerably more involved.

Comment on "Calculation of Turbulent Boundary-Layer Shock-Wave Interaction"

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WILCOX[1] recently reported on a numerical calculation scheme which incorporates a two-parameter turbulence model with finite-difference techniques. Predictions generated by this method, for the two-dimensional viscous-inviscid interaction of an incident oblique shock wave with a turbulent boundary layer, were presented and comparisons with the experimental data of Ref. 2 were shown.†

It should be noted that some of these experimentally determined quantities were incorrectly listed, and/or referred to, in the comparisons made by Wilcox. It is the purpose of this Comment to show the predictions of Ref. 1 in comparison with the combined results of Refs. 2 and 3, and to briefly reiterate the conclusions of this earlier study. At the outset, it should be restated that neither flowfield, as generated by the two distinct shock generator systems of Refs. 2 and 3, was strictly two dimensional; any comparison of these results with the predictions of a mathematically two-dimensional calculation scheme must be viewed with this fact in mind.

Experimentally determined pressure rises to separation and reattachment, upstream influence length, and length of separated flow, as a function of measured over-all pressure rise, are shown in Figs. 1–3; Fig. 4 shows sonic line locations throughout the interaction regions of those four incident shock strength flowfields probed. Measured over-all pressure rises for incipient separation, and experimentally determined separation and reattachment point locations, are also included. Recall that G1 and G2 refer to the two shock generator systems used; G1 was a full-span wedge, while G2 was full span between two side plates, used to cut off the channel sidewall boundary layers. (All nomenclature is as previously listed.)

As stated in Ref. 1, the calculation scheme described did predict the qualitative structure of the interaction flowfield (e.g., existence of both separation and reattachment shocks, a region of reverse flow, and the general shape/behavior of the

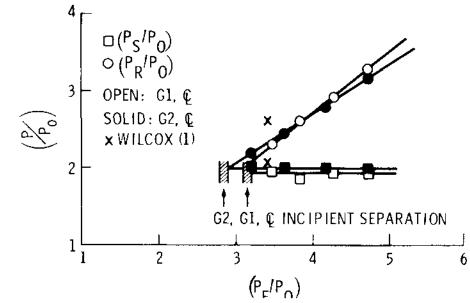

Fig. 1 Pressure rise to separation and reattachment vs over-all pressure rise.

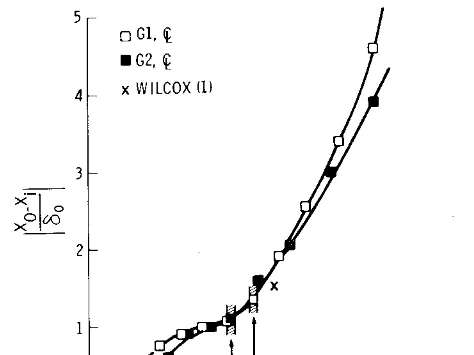

Fig. 2 Upstream influence length vs over-all pressure rise.

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† Experimental results presented in this discussion were generated by the author during his N.R.C. Postdoctoral Research Associateship at NASA Ames Research Center, Moffett Field, Calif.